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WEAK TYPE (1,1) ESTIMATES FOR SOME INTEGRAL OPERATORS RELATED TO ROUGH MAXIMAL FUNCTIONS*

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ABSTRACT

We study the problem of determining for which integrable functions $G\colon \mathbf{R} \to (0,\infty)$ the operator $f \to \frac{1}{y}G(y.) * f(x)$, which maps functions on the real line into functions defined on the upper half-plane $\mathbf{R}^2_+$, is of weak type (1,1). Here, $\mathbf{R}^2_+$ is endowed with the measure $y\,dx\,dy$. The conditions we will impose are related to the distribution of the mass of G.

One of the motivations for this study comes from the problem of deciding whether there is a weak type (1,1) inequality for the "rough" modification of the standard maximal function, obtained by inserting in the mean values a factor Ω which depends only on the angle. Here, $\Omega \geq 0$ is any integrable function on the sphere. Our estimates for the first-mentioned problem allow us to answer in the affirmative, the second one in dimension two, when we restrict the operator to radial functions. Some extensions to higher dimensions in the context of both problems are also discussed.

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1. Introduction and statements of results

The purpose of this work is to study the behaviour of certain "averaging" operators which naturally arise as integral operators from functions defined, say, in $\mathbf{R}$ into functions defined in the half-space $\mathbf{R}^2_+$ endowed with an appropriate measure. To be more precise, given an integrable function G, we will describe sufficient conditions on G to ensure the boundedness of the linear operator

$$f \to \frac{1}{y} G(y.) * f(x),$$

from $L^1(\mathbf{R})$ into $L^{1,\infty}(\mathbf{R}^2_+, ydxdy)$.

These questions came from the following problem. Does the modification of the Hardy–Littlewood maximal operator by means of a "rough", homogeneous function satisfy a weak-type (1,1) estimate, under the sole condition that the rough function be integrable when restricted to the unit sphere? We will prove that this estimate holds in two dimensions when the operator is applied to radial functions only, by reducing this problem to the boundedness of certain integral operators like the one mentioned above.

We begin by stating the following particular result, which represents a basic tool in the development of our study:

THEOREM 1: *Let h be a positive, even function in $L^1(\mathbf{R})$ which is decreasing on $(0,\infty)$. Then the operator*

$$Tf(x,y) = \frac{1}{y} h(y.) * f(x)$$

is bounded from $L^1(\mathbf{R})$ into $L^{1,\infty}(\mathbf{R}^2_+, ydxdy)$, with a constant which only depends on $||h||_1$.

Next, we will replace h by a more general function called G. This G will still verify a weak smoothness condition expressed in terms of the distribution of its masses and the distance to the origin. The conclusion is similar to that of Theorem 1.

THEOREM 2: *Given $\epsilon > 0$ and a sequence of real numbers $\{a_k\}_{-\infty}^{\infty} \in l^1$, we define*

$$G(x) = G^\epsilon(x) = \sum_{k=-\infty}^{\infty} a_k \frac{1}{|J_k|} \chi_{J_k}(x),$$

where J_k denotes the interval $((1+\epsilon)^{-k},(1+\epsilon)^{1-k}]$. Then, the operator T defined on functions $f \in L^1(\mathbf{R})$ by

$$Tf(x,y) = T^\epsilon f(x,y) = \frac{1}{y} G(y.) * f(x)$$

satisfies the weak type estimate

$$\mu(\{(x,y) \in \mathbf{R}^2_+ \colon |Tf(x,y)| > \lambda\}) \leq \frac{C_\epsilon}{\lambda} \left(\sum_{k=-\infty}^{\infty} |a_k| \right) ||f||_1, \ \lambda > 0,$$

where $d\mu = ydxdy$ and $C_\epsilon = O((\log \frac{1}{\epsilon})^2)$ as $\epsilon \to 0$.

As an application of these results, we consider a maximal operator in $\mathbf{R}^n$ associated with a positive function Ω, homogeneous of degree 0 and integrable on $\mathbf{S}^{n-1}$. Define

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} \Omega(y)|f(x-y)|dy.$$

To determine whether or not M_Ω is bounded from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}^n)$ has become an important open problem in harmonic analysis. In 1975, R. Fefferman [F] proved this result under the stronger condition that Ω be of "finite entropy", and this was extended by F. Soria in [So] to the case in which Ω is in the (larger) block space B_∞.

Later on, M. Christ [Ch] proved the result for $n = 2$ under the condition $\Omega \in L^q(S^1)$, $q > 1$. This was then improved by M. Christ and J.L. Rubio de Francia [Ch-R] to arbitrary n and $\Omega \in L \log L(S^{n-1})$.

It is worth pointing out that any function of finite entropy belongs to $L \log L$, whereas B_∞ has the property that the smallest rearrangement invariant space B of functions which contains it, also contains $L \log \log L$. Moreover, the only Orlicz space which contains B, and is contained in L^1, is L^1 (see [So]).

Other recent results regarding the boundedness of M_Ω are due to S. Hofmann [Ho], who obtains weighted inequalities for power weights, and to S. Hudson [Hu], who gives a proof for the case $\Omega \in L^1(S^1)$ and Ω decreasing.

Let, for $\omega \in S^{n-1}$,

$$M_\omega f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x-t\omega)|dt$$

be the one-dimensional Hardy–Littlewood maximal operator along the direction ω. To estimate M_Ω in terms of M_ω, we write the integral in the definition of $M_\Omega f$ in polar coordinates and interchange the order of the supremum and the angular integration. It easily follows that M_Ω is dominated by the operator M_Ω^* defined by

$$M_\Omega^* f(x) = \int_{S^{n-1}} \Omega(\omega) M_\omega f(x) d\omega.$$

Here and in the sequel, $d\omega$ denotes the area measure of S^{n-1}.

There is no weak type (1,1) estimate for M_Ω^* in general. This is a consequence of an example due to R. Fefferman [F], p. 176. With $\Omega = 1$, he proves that M_Ω^* is not of weak type (1,1).

We now restrict ourselves to the case in which f is a radial function. The situation in this case obviously becomes much simpler, since the geometry of radial functions is easier to handle. Observe, however, that $M_\Omega f$ and $M_\Omega^* f$ are not radial even though f is. It is no surprise that in two dimensions M_Ω turns out to be of weak type (1,1) when restricted to radial functions. What is maybe a little surprising is that the same holds for $M_\Omega^* f$, as follows.

THEOREM 3: *There exists an absolute constant C such that for any Ω with*

$$\int_{S^1} \Omega(\omega) d\omega = 1,$$

and any radial function $f \in L^1(\mathbf{R}^2)$, one has

$$|\{x \in \mathbf{R}^2 : M_\Omega^* f(x) > \lambda\}| \le C \frac{1}{\lambda} \|f\|_1, \quad \lambda > 0.$$

In a forthcoming paper, the authors extend this theorem to $\mathbf{R}^n$, $n > 2$, and prove an analogous result for rough singular integral operators.

The paper is organized as follows. We present in Section 2 the proofs of Theorems 1 and 2. Section 3 is devoted to the proof of Theorem 3. Finally, Section 4 contains some additional extensions of these results and some open questions related to the subject.

By C, we shall denote various constants.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1: By a limiting argument, we may assume, without loss of generality, that h has compact support and then, by dilation invariance, that its support lies within some small interval, say $[-\frac{1}{8}, \frac{1}{8}]$.

We may also assume that f is positive and that h has the form

$$h = \sum_{k\geq 0} a_k 2^k \chi_{(-2^{-k-3}, 2^{-k-3})}, \quad a_k \geq 0,$$

so that $||h||_1 \sim \sum_k a_k$.

Define for a fixed $\lambda > 0$ and each $y > 0$

$$A_\lambda(y) = |\{x \in \mathbf{R}\colon \frac{1}{y} h(y.) * f(x) > \lambda\}|,$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbf{R}$. Writing $d\mu = ydxdy$, we have

$$\mu\{(x,y) \in \mathbf{R}^2_+\colon Tf(x,y) > \lambda\} = \int_0^\infty A_\lambda(y) y dy.$$

Observe that the fact that h is even and decreasing on $(0, \infty)$ implies that the function $A_\lambda(y)$ decreases as y increases. Hence, the above integral can be controlled by the sum

$$\sum_{j\in\mathbf{Z}} A_\lambda(2^j) 2^{2j}.$$

Obviously, it will suffice to show that for a given $j_0 \in \mathbf{Z}$,

$$\sum_{j\geq j_0} A_\lambda(2^j) 2^{2j} \leq \frac{C}{\lambda} ||h||_1 ||f||_1,$$

with constant C independent of j_0.

Let us introduce the intervals

$$I_l^i = (2^{-l} i, 2^{-l}(i+1)], \quad l, i \in \mathbf{Z}.$$

The conditional expectation of a function $g \in L^1_{loc}$ at level l is defined as

$$E_l g(x) = \sum_{i\in\mathbf{Z}} \left(\frac{1}{|I_l^i|} \int_{I_l^i} g(u) du \right) \chi_{I_l^i}(x).$$

Now, for any $s \in \mathbf{R}$, we have

$$2^{-j}h(2^j.) * f(x) = 2^{-2j}\sum_{k\geq 0} a_k 2^{k+j}\int_{x-2^{-(j+k)-3}}^{x+2^{-(j+k)-3}} f(u)du$$
$$= 2^{-2j}\sum_{k\geq 0} a_k 2^{k+j}\int_{x+s-2^{-(j+k)-3}}^{x+s+2^{-(j+k)-3}} f(u-s)du.$$

We will need the following simple result, which will be proved later.

CLAIM: *For each $x \in \mathbf{R}$, $k \geq 0$, and $j \geq j_0$, one of the two intervals $I_x = (x - 2^{-(j+k)-3}, x + 2^{-(j+k)-3})$ and $I'_x = 2^{-j_0}/3 + I_x$ lies entirely within some interval I^i_{j+k}.*

It follows easily from the claim and the above inequalities that

$$Tf(x, 2^j) \leq 2^{-2j}\sum_{k\geq 0} a_k E_{j+k}f(x) + 2^{-2j}\sum_{k\geq 0} a_k E_{j+k}\hat{f}\left(x + \frac{2^{-j_0}}{3}\right),$$

where $\hat{f}(u) = f(u - \frac{2^{-j_0}}{3})$. Obviously, $\hat{f}$ depends on j_0, but $||\hat{f}||_1$ does not. Thus, the problem is equivalent to showing the existence of a universal constant C such that for all $j_0 \in \mathbf{Z}$, $\lambda > 0$,

$$\sum_{j\geq j_0} |\{x \in \mathbf{R}\colon \sum_{k\geq 0} 2^{-2j}a_k E_{j+k}f(x) > \lambda\}|2^{2j} \leq \frac{C}{\lambda}\sum_{0}^{\infty} a_k||f||_1. \tag{1}$$

Roughly speaking, the idea of the proof is as follows. For each j, Chebyshev's inequality implies

$$|\{x\colon \sum_{k\geq 0} 2^{-2j}a_k E_{j+k}f(x) > \lambda\}|2^{2j} \leq \frac{1}{\lambda}\sum_{0}^{\infty} a_k||f||_1. \tag{2}$$

To be able to sum in j, we must show that this inequality can be improved for most j. Indeed, we shall see that in order to reach level λ, the sums

$$\sum_{k\geq 0} 2^{-2j}a_k E_{j+k}f,$$

occurring in (2), must "use" different parts of f for different j. This would amount to decomposing f as $f = \sum f_j$ and verifying (2) with, in the right-hand side, the L^1 norm of f_j. Then, one could sum in j. In fact, our argument is slightly more complicated. We choose one decomposition of j for each value of k, i.e., for each part of h, as follows:

LEMMA A: *There exists a double sequence of positive functions* $\{f_{j,k}\}_{k\geq 0, j\geq j_0}$ *such that*

$$\sum_{j\geq j_0} f_{j,k} \leq f, \quad \text{for each } k \geq 0, \tag{3}$$

and

$$\{x \in \mathbf{R}\colon \sum_{k\geq 0} 2^{-2j} a_k E_{j+k} f(x) > \lambda\} \subset \{x \in \mathbf{R}\colon \sum_{k\geq 0} 2^{-2j} a_k E_{j+k} f_{j,k}(x) \geq \lambda/2\}. \tag{4}$$

Assuming the validity of Lemma A, we observe that, by Chebyshev's inequality, the left-hand side of (1) is majorized by

$$\frac{2}{\lambda}\sum_{j\geq j_0}\sum_{k} a_k 2^{-2j}\left(\int_{\mathbf{R}} E_{j+k} f_{j,k}(x)dx\right)2^{2j}$$
$$= \frac{2}{\lambda}\sum_{k} a_k \sum_{j\geq j_0}\int_{\mathbf{R}} f_{j,k}(x)dx \leq \frac{2}{\lambda}\left(\sum_{k} a_k\right)\left(\int f(x)dx\right),$$

so that (1) follows.

The proof of Lemma A is by induction and relies on the following:

LEMMA B: *Given* $\alpha > 0$ *and a sequence of positive measurable functions* $\{F_n\}_{n\geq n_0}$ *on* $\mathbf{R}$, *we can find another sequence of functions* $\{G_n\}_{n\geq n_0}$, *with* $0 \leq G_n \leq F_n$ *for* $n \geq n_0$, *and such that*

$$\sum_n E_n G_n = \min(\alpha, \sum_n E_n F_n). \tag{5}$$

Proof: We shall prove something slightly stronger than (5), namely,

$$E_n G_n = \min\left(\alpha, \sum_{m=n_0}^{n} E_m F_m\right) - \sum_{m=n_0}^{n-1} E_m G_m, \quad n \geq n_0. \tag{6}$$

Let us start with a trivial observation: If I is an interval, F is a positive function on I, and we choose $0 \leq \beta \leq \frac{1}{|I|}\int_I F$, then there exists $0 \leq G \leq F$ such that

$$\frac{1}{|I|}\int_I G = \beta.$$

This allows us immediately to find some $0 \leq G_{n_0} \leq F_{n_0}$ satisfying the relation

$$E_{n_0} G_{n_0} = \min(\alpha, E_{n_0} F_{n_0}).$$

Now, suppose that we have inductively constructed functions $G_{n_0},\dots,G_{l-1}$ with $0\le G_n\le F_n$, satisfying (6) for each $n_0\le n\le l-1$. To find G_l, notice that the case $n=l-1$ of (6) implies that the expression in the right-hand side of (6) for $n=l$ is nonnegative and dominated by E_lF_l. Since the same expression is also constant on each interval I_l^i, the above observation applies again and produces the desired G_l. This completes the induction process and, hence, the proof of Lemma B. ∎

Proof of Lemma A: We first apply Lemma B with $\alpha=\lambda/2$, $n_0=j_0$, and $F_{j_0+k}=2^{-2j_0}a_kf$, $k=0,1,2,\dots$. This gives functions $f_{j_0,k}\le f$ such that

$$\sum_{k\ge 0}E_{j_0+k}(2^{-2j_0}a_kf_{j_0,k})=\min\left(\lambda/2,\sum_{k\ge 0}E_{j_0+k}(2^{-2j_0}a_kf)\right).$$

Assume that we have found $\{f_{l,k}\}_{k\ge 0}$, $l=j_0,j_0+1,\dots,j-1$ with $\sum_{l=j_0}^{j-1}f_{l,k}\le f$, for each k, and

$$\sum_{k\ge 0}E_{l+k}(2^{-2l}a_kf_{l,k})=\min\left(\lambda/2,\sum_{k\ge 0}E_{l+k}(2^{-2l}a_k[f-\sum_{i=j_0}^{l-1}f_{i,k}])\right),$$

for each $l\le j-1$. Now, we apply Lemma B again, with $\alpha=\lambda/2$, $n_0=j$, and

$$F_{j+k}=2^{-2j}a_k\left(f-\sum_{l=j_0}^{j-1}f_{l,k}\right),\quad k=0,1,2,\dots.$$

This produces functions $f_{j,k}$, $k\ge 0$, such that

$$f_{j,k}\le f-\sum_{l=j_0}^{j-1}f_{l,k},\quad k\ge 0, \tag{7}$$

and

$$\begin{aligned}&\sum_{k\ge 0}E_{j+k}(2^{-2j}a_kf_{j,k})\\ &=\min\left(\lambda/2,\sum_{k\ge 0}E_{j+k}(2^{-2j}a_kf)-\sum_{k\ge 0}E_{j+k}(2^{-2j}a_k\sum_{l=j_0}^{j-1}f_{l,k})\right).\end{aligned} \tag{8}$$

By induction in j, we thus obtain a double sequence $f_{j,k}$, $j\ge j_0,k\ge 0$. Conclusion (3) of Lemma A follows directly from (7). To verify (4), we first observe the simple inequality $E_j g \le 2^{j-l} E_l g$ for $l < j$ and any $g \ge 0$. This allows us to estimate the last term in (8); indeed

$$\begin{aligned}\sum_{k\ge 0} E_{j+k}\left(2^{-2j} a_k \sum_{l=j_0}^{j-1} f_{l,k}\right) &\le \sum_{k\ge 0} 2^{-2j} a_k \sum_{l=j_0}^{j-1} 2^{j-l} E_{l+k} f_{l,k} \\ &= \sum_{l=j_0}^{j-1} 2^{l-j} \sum_{k\ge 0} 2^{-2l} a_k E_{l+k} f_{l,k} \le \lambda/2,\end{aligned}$$

the last inequality because of (8). Now, (4) is immediate from this and (8). Lemma A is proved. ∎

For the conclusion of Theorem 1, it only remains to give a

Proof of the Claim: Set $\tau = j + k$. Assume I_x intersects both I_τ^{i-1} and I_τ^i for some $i \in \mathbf{Z}$. Let l be the (unique) integer such that

$$l 2^{-\tau} < \frac{2^{-j_0}}{3} \le (l+1) 2^{-\tau}.$$

We make here the important (and trivial) observation that since $3l \le 2^{\tau - j_0} \le 3(l+1)$ and $\tau - j_0 \in \mathbf{N}$ by hypothesis, we must have either $2^{\tau-j_0} = 3l+1$ or $2^{\tau-j_0} = 3l+2$. In both cases, we obtain

$$\left(l + \frac{1}{3}\right) 2^{-\tau} \le \frac{2^{-j_0}}{3} \le \left(l + \frac{2}{3}\right) 2^{-\tau}.$$

It is now easy to see that $I'_x \subset I_\tau^{l+i}$. ∎

Proof of Theorem 2: We can assume, with no loss of generality, that $1+\epsilon = 2^{2^{-\nu}}$, for some $\nu \in \{1, 2, 3, \dots\}$. Note that the mean value theorem implies

$$2^{-1-\nu} < 2^{2^{-\nu}} - 1 < 2^{-\nu}. \tag{9}$$

We decompose G as

$$G(x) = \sum_{\kappa=0}^{2^{1+\nu}\nu - 1} \sum_{k = l2^{1+\nu}\nu + \kappa,\, l \in \mathbf{Z}} a_k \frac{1}{|J_k|} \chi_{J_k} = \sum_{\kappa=0}^{2^{1+\nu}\nu-1} G_\kappa(x),$$

say.

From the Stein–N.Weiss adding-up lemma (see [S-NW]), it follows that we need only show that each operator

$$T_\kappa f(x,y) = \frac{1}{y} G_\kappa(y.) * f(x), \quad \kappa = 0,\dots,2^{1+\nu}\nu - 1,$$

is bounded from $L^1(\mathbf{R})$ into $L^{1,\infty}(\mathbf{R}^2_+, d\mu)$, with a constant at most $C\nu\|G_\kappa\|_1$, where C is a universal constant. Observe that

$$(1+\epsilon)^{-\kappa} G_\kappa((1+\epsilon)^{-\kappa}.) = \sum_{l\in\mathbf{Z}} a_{l2^{1+\nu}\nu+\kappa} \frac{2^{2\nu l}}{\epsilon} \chi_{(2^{-2\nu l},(1+\epsilon)2^{-2\nu l}]},$$

and so, because of dilation invariance, we need only consider the case $\kappa = 0$.

Given $\lambda > 0$, let us define as before

$$A_\lambda(y) = |\{x \in \mathbf{R}\colon |T_0 f(x,y)| > \lambda\}|.$$

Then,

$$\int_0^\infty A_\lambda(y) y\,dy = \sum_j \int_{2^{2\nu j}}^{2^{2\nu(j+1)}} A_\lambda(y) y\,dy = \int_1^{2^{2\nu}} \left(\sum_j A_\lambda(2^{2\nu j}\xi) 2^{4\nu j}\xi^2\right) \frac{d\xi}{\xi}.$$

Thus, it suffices to show that there exists a universal constant C such that for every $\xi \in [1, 2^{2\nu}]$ one has

$$\sum_j A_\lambda(2^{2\nu j}\xi) 2^{4\nu j}\xi^2 \le \frac{C}{\lambda} \|G_0\|_1 \|f\|_1.$$

By rescaling, we may assume that $\xi = 1$.

For convenience, we change notation slightly. Write

$$G = \sum_l a_l |J_l|^{-1} \chi_{J_l},$$

where now $J_l = (2^{-2\nu l}, 2^{2^{-\nu}} 2^{-2\nu l}]$ and we write a_l instead of $a_{l2^{1+\nu}\nu}$. What we must show is that

$$\text{(10)} \qquad \sum_j |\{x \in \mathbf{R}\colon |2^{-2\nu j} G(2^{2\nu j}.) * f(x)| > \lambda\}| 2^{4\nu j} \le \frac{C}{\lambda} \sum_l |a_l| \, \|f\|_1,$$

with C independent of $\nu \in \mathbf{N}$, $f \in L^1(\mathbf{R})$, and the sequence $\{a_k\} \in l^1$. We may obviously assume that $f \ge 0$ and $a_k \ge 0$ for each k. In this proof, we set

$$I_k^i = (i2^{-\nu(2k+1)}, (i+1)2^{-\nu(2k+1)}], \quad k, i \in \mathbf{Z}.$$

Define, for a given $N \in \mathbf{N}$, the conditional expectation at level k "displaced by N" as

$$E_k^N g(x) = \sum_{i\in\mathbf{Z}} \left(\frac{1}{|I_k^{i-N}|} \int_{I_k^{i-N}} g(u)du \right) \chi_{I_k^i}(x) = E_k^0 g(x - N2^{-\nu(2k+1)}).$$

We notice that (9) implies that if $x \in I_k^i$, then $x - J_k \subset \bigcup_{N=2^\nu}^{2^\nu+1} I_k^{i-N}$. Hence,

$$\begin{aligned} 2^{-2\nu j} G(2^{2\nu j}.) * f(x) &= 2^{-4\nu j} \sum_l a_l \frac{1}{|J_{l+j}|} \chi_{J_{l+j}} * f(x) \\ &\leq \sum_{N=2^\nu}^{2^\nu+1} 2^{-4\nu j} \sum_l a_l E_{l+j}^N f(x). \end{aligned}$$

We fix $N \in \{2^\nu, 2^\nu + 1\}$. In order to prove (10), it suffices to show that for each $l_0, j_0 \in \mathbf{Z}$

$$(11) \qquad \sum_{j\geq j_0} |\{x \in \mathbf{R}\colon \sum_{l\geq l_0} 2^{-4\nu j} a_l E_{l+j}^N f(x) > \lambda\}| 2^{4\nu j} \leq \frac{C}{\lambda} \sum_l a_l \, ||f||_1,$$

with constant C independent of j_0, l_0.

The idea of the proof is to reduce the problem to the case in which G is a decreasing function on $(0, \infty)$, and then apply Theorem 1. We start by defining the operator $\hat{E}_k^N f(x)$, for $x \in I_k^i$, as $E_k^N f(x)$ provided both I_k^i and I_k^{i-N} are subsets of the same I_{k-1}^j, and $\hat{E}_k^N f(x) = 0$ otherwise.

LEMMA C: *With the notation above, there exists for every $k_0 \in \mathbf{Z}$ a bijection $\pi\colon \mathbf{R} \to \mathbf{R}$, preserving Lebesgue measure, and such that for $k \geq k_0$,*

$$(12) \qquad \hat{E}_k^N f(x) \leq E_k^1 (f \circ \pi^{-1})(\pi(x)).$$

With $k_0 = l_0 + j_0$, this lemma implies for $j \geq j_0$ that

$$\begin{aligned} &\sum_{l\geq l_0} 2^{-4\nu j} a_l \hat{E}_{l+j}^N f(x) \\ &\leq \sum_{l\geq l_0} 2^{-4\nu j} a_l E_{l+j}^1 (f \circ \pi^{-1})(\pi(x)) \\ &\leq \sum_{l\geq l_0} 2^{-4\nu j} a_l [\frac{1}{2^{-\nu} 2^{-2\nu(l+j)}} \chi_{(0, 2^{-2\nu(l+j)-\nu+1}]} * (f \circ \pi^{-1})](\pi(x)) \\ &= [2^{-2\nu j} G^*(2^{2\nu j}.) * (f \circ \pi^{-1})](\pi(x)), \end{aligned}$$

where

$$G^*(x) = 2\sum_{l\geq l_0} a_l \frac{1}{2^{-2\nu l-\nu+1}} \chi_{(0,2^{-2\nu l-\nu+1}]}.$$

Hence, if we set $A^*_\lambda(y) = |\{x: \frac{1}{y}G^*(y.) * (f\,o\,\pi^{-1})(x) > \lambda\}|$, the left-hand side of (11), with E^N_k replaced by $\hat{E}^N_k$, is majorized by

$$\sum_{j\geq j_0} A^*_\lambda(2^{2\nu j})2^{4\nu j}.$$

Since $A^*_\lambda(y)$ is decreasing in y, this sum is at most

$$C\int_0^\infty A^*_\lambda(y)ydy = C\mu(\{(x,y)\in \mathbf{R}^2_+: \frac{1}{y}G^*(y.) * (f\,o\,\pi^{-1})(x) > \lambda\}).$$

Because of Theorem 1, this is dominated by

$$C\frac{1}{\lambda}||G^*||_1\,||f\,o\,\pi^{-1}||_1 \leq C\frac{1}{\lambda}(\sum_l a_l)\,||f||_1.$$

Observe that the constants C here are independent of π and, therefore, of j_0 and l_0.

This takes care of that part of the left-hand side in (11) corresponding to intervals I^i_k and I^{i-N}_k in the same I^j_{k-1}. The remaining part can be reduced to this by a shift argument similar to the one used in the proof of Theorem 1 (see the claim). This, together with the proof of Lemma C, given below, finishes the proof of Theorem 2. ∎

Proof of Lemma C: We shall construct π so that each image $\pi(I^i_k)$ is some $I^{i'}_k$, for all $k \geq k_0$. Inequality (12) will follow if we can show that whenever I^i_k and I^{i-N}_k, $k \geq k_0$, are in the same I^j_{k-1}, they are mapped onto adjacent intervals in the same order; i.e., with the previous notation, $i' = (i-N)'+1$. We shall define π as an infinite composition product $\pi = \Pi^\infty_{k=k_0}\psi_k$ of commuting bijections ψ_k. Each I^i_k is contained in some I^j_{k-1}, and ψ_k will map I^i_k by translation onto some $I^{i''}_k$ contained in the same I^j_{k-1}. In particular, ψ_k will map each $I^i_{k'}$ onto itself for $k' < k$ and will move each I^i_k rigidly. Fix $k \geq k_0$. It is easy to find a ψ_k of this type such that (12) holds with π replaced by ψ_k. Indeed, we need only find an appropriate permutation of those I^i_k contained in each I^j_{k-1}, for instance as follows: The set of these I^i_k has a unique partition into maximal subsets of the type

$$\{I^i_k, I^{i+N}_k, I^{i+2N}_k, \ldots, I^{i+\mu N}_k\}.$$

It is enough to take a permutation which sends the elements of each such subset onto consecutive intervals in the same order.

The same permutation is used in each I^j_{k-1}. This gives a map ψ_k satisfying (12); indeed, equality holds in (12) at "most points". It is easily seen that the ψ_k commute. The infinite product $\pi = \Pi\psi_k$ exists because $(\psi_{k_0} o \ldots o \psi_n(x))_{n \geq k_0}$ is a Cauchy sequence for each x. Further, π is bijective since it has an inverse, which turns out to be of the same type. Considering intervals I^i_k, one can see that π preserves Lebesgue measure.

Finally, we claim that π, like ψ_k, satisfies (12). When we compose ψ_k with those $\psi_{k'}$ with $k' > k$, then the effect is that points are moved within each I^i_k. The mean values appearing in (12) are, thus, not changed. Those $\psi_{k'}$ with $k' < k$ move each I^j_{k-1} rigidly to some other position, and so (12) is seen to hold for π.

The lemma is proved. ∎

3. Boundedness of rough operators

Proof of Theorem 3: Although $n = 2$ in this theorem, we write part of the proof for a general n.

Given $\omega \in \mathbf{S}^{n-1}$ and a radial function $f \in L^1$, the behaviour of $M_\omega f$ can be described in terms of the action of simple operators on the radial projection, f_0, of f, i.e., $f(x) = f_0(|x|)$. Define one-sided maximal operators on the positive half-line by

$$M_{\rightarrow} g(r) = \sup_{h>0} \frac{1}{h} \int_r^{r+h} |g(t)|dt,$$

$M_{\leftarrow}$ analogously, and

$$\hat{M}_{\leftarrow} g(r) = \sup_{0<h<r/2} \frac{1}{h} \int_{r-h}^{r} |g(t)|dt.$$

Let $A(x,\omega) \in [0,\pi]$ denote the angle between x and ω, and set $\epsilon_\omega(x) = \sin A(x,\omega)$. Observe that $\epsilon_\omega(x) = \epsilon_\omega(x')$, with $x' = x/|x| \in \mathbf{S}^{n-1}$. From [C-H-S], we know that for $x \in \mathbf{R}^n$

$$M_\omega f(x) \leq C M_{\rightarrow} f_0(|x|),$$

when $A(x,\omega) \geq \pi/2$. When $A(x,\omega) \leq \pi/2$, we notice that the point on the line $\{x - t\omega\colon t \in \mathbf{R}\}$ which is closest to the origin corresponds to $t = L_\omega(x) =$

$|x|(1-\epsilon_\omega(x)^2)^{1/2}$. Introduce the operator

$$\begin{aligned} A_\omega g(x) &= \frac{1}{L_\omega(x)} \int_0^{L_\omega(x)} g(|x-t\omega|)dt \\ &= \frac{1}{|x|(1-\epsilon_\omega(x)^2)^{1/2}} \int_{|x|\epsilon_\omega(x)}^{|x|} g(t) \frac{t}{(t^2-(|x|\epsilon_\omega(x))^2)^{1/2}} dt. \end{aligned}$$

It now follows from [C-H-S] that for $A(x,\omega) \le \pi/2$

$$M_\omega f(x) \le C(M_{\rightarrow} f_0(|x|) + \hat{M}_{\leftarrow} f_0(|x|) + A_\omega f_0(x)).$$

One easily sees that $\hat{M}_{\leftarrow}$ is bounded from $L^1(r^{n-1}dr)$ into $L^{1,\infty}(r^{n-1}dr)$, dividing the half-line into dyadic subintervals. Also, $W(r) = r^{n-1}$ satisfies $M_{\leftarrow} W \le W$ (in fact $M_{\leftarrow} W = W$.) But then W is an A_1 weight for $M_{\rightarrow}$, as proved by Martin-Reyes, Ortega and de la Torre [MOT, Theorem 1]. Hence, we need only show that the operator

$$A_\Omega g(x) = \int_{A(x,\omega)\le\pi/2} \Omega(\omega) A_\omega g(x) d\omega$$

maps $L^1(r^{n-1}dr)$ into $L^{1,\infty}(\mathbf{R}^n)$.

Let us split the above integral into two pieces, $\int_1$ and $\int_2$, according to whether $A(x,\omega) \le \pi/4$ or $\pi/4 < A(x,\omega) \le \pi/2$, respectively. The boundedness properties of the second part are rather good, since one has a strong type estimate. Indeed,

$$\begin{aligned} &\left\| \int_2 \Omega(\omega) A_\omega g d\omega \right\|_{L^1(\mathbf{R}^n)} \\ &= \int_{S^{n-1}} dx' \int_0^\infty \int_2 \Omega(\omega) \frac{d\omega}{r(1-\epsilon_\omega(x')^2)^{1/2}} \int_{r\epsilon_\omega(x')}^r \frac{g(t)\,t\,dt}{(t^2-(r\epsilon_\omega(x'))^2)^{1/2}} r^{n-1}\,dr \\ &\le \int_{S^{n-1}} dx' \int_2 \Omega(\omega) \frac{d\omega}{(1-\epsilon_\omega(x')^2)^{1/2}} \int_0^\infty g(t)\, t^{1/2} \left[\int_t^{t/\epsilon_\omega(x')} \frac{r^{n-2}\,dr}{(t-r\epsilon_\omega(x'))^{1/2}} \right] dt \\ &\le C \int_{S^{n-1}} dx' \int_2 \Omega(\omega) \frac{d\omega}{(1-\epsilon_\omega(x'))^{1/2}} \int_0^\infty g(t) t \left[\frac{t}{\epsilon_\omega(x')} \right]^{n-2} \frac{(1-\epsilon_\omega(x'))^{1/2}}{\epsilon_\omega(x')} dt \\ &= \left[\int_{S^{n-1}} dx' \int_2 \Omega(\omega) \frac{d\omega}{(\epsilon_\omega(x'))^{n-1}} \right] \left[\int_0^\infty g(t) t^{n-1}\,dt \right] \le C \int_0^\infty g(t)\, t^{n-1}\,dt. \end{aligned}$$

The last estimate follows since $\epsilon_\omega(x') > \sin\pi/4$ here.

For the integrand in $\int_1$ we have

$$L_\omega(x) = |x|(1-\epsilon_\omega(x)^2)^{1/2} \sim |x|.$$

From now on, we assume $n = 2$. Write x and ω in polar coordinates as $x = re^{i\theta}$, $\omega = e^{i\phi}$, with $\theta, \phi \in (-\pi, \pi]$ and set $\Omega(\phi) = \Omega(e^{i\phi})$, by a slight abuse of notation. Then,

$$\int_1 \Omega(\omega)A_\omega g(x)d\omega \le C\int_1 \Omega(\phi)\,d\phi\frac{1}{r}\int_{r|\sin(\phi-\theta)|}^{\infty} g(t)\frac{t^{1/2}}{(t-r|\sin(\phi-\theta)|)^{1/2}}dt.$$

Here, the integral in ϕ is taken over those ϕ of distance at most $\pi/4$ from θ. This distance is to be taken along the unit circle, but to solve our problem, we may use instead the distance of the real line. Set

$$G(u) = \int_{|u|}^{\infty} g(t)\frac{t^{1/2}}{(t-|u|)^{1/2}}dt.$$

What we then need to show is that

$$\int_{|\phi-\theta|\le\pi/4} \Omega(\phi)\frac{1}{r}G(r\sin(\phi-\theta))\,d\phi \in L^{1,\infty}([-2\pi, 2\pi]\times(0,\infty), rd\theta dr),$$

with quasi-norm bounded by $C||g||_{L^1(rdr)}$. Let us make the observation that

$$\frac{1}{r}G(r\sin u) - \frac{1}{r}G(ru) \in L^1((-\pi/4,\pi/4)\times(0,\infty), rdudr).$$

In fact,

$$\begin{aligned}
&\int_{-\pi/4}^{\pi/4}\int_0^{\infty} |\frac{1}{r}G(r\sin u) - \frac{1}{r}G(ru)|\,r\,dr\,du\\
&\le 2\int_0^{\pi/4}\int_0^{\infty}\Big[\int_{r\sin u}^{ru} g(t)t^{1/2}\frac{1}{(t-r\sin u)^{1/2}}dt\\
&\quad+\int_{ru}^{\infty} g(t)t^{1/2}|\frac{1}{(t-r\sin u)^{1/2}}-\frac{1}{(t-ru)^{1/2}}|dt\Big]\,dr\,du\\
&= 2\int_0^{\pi/4}\int_0^{\infty} g(t)t^{1/2}\Big[\int_{t/u}^{t/\sin u}\frac{1}{(t-r\sin u)^{1/2}}dr\\
&\quad+\int_0^{t/u}[\frac{1}{(t-ru)^{1/2}}-\frac{1}{(t-r\sin u)^{1/2}}]dr\Big]\,dt\,du\\
&= 4\int_0^{\infty} g(t)t\,dt\int_0^{\pi/4}[\frac{2(u-\sin u)^{1/2}}{u^{1/2}\sin u}+\frac{1}{u}-\frac{1}{\sin u}]du\\
&= C||g||_{L^1(rdr)}\int_0^{\pi/4} V(u)du,
\end{aligned}$$

and we notice that the quantity $V(u)$ thus defined is bounded in $(0, \pi/4)$.

The proof is now reduced to showing that

$$\int_{|\phi-\theta|\leq\pi/4} \Omega(\phi)\frac{1}{r}G(r(\phi-\theta))\,d\phi \in L^{1,\infty}([-2\pi, 2\pi]\times(0,\infty), rd\theta dr),$$

with a bound for the quasi-norm depending only on $||g||_{L^1(rdr)}$. This will be a consequence of the inequality

$$\left\|\frac{1}{r}G(r.)*f(\theta)\right\|_{L^{1,\infty}(\mathbf{R}^2_+, rd\theta dr)} \leq C||g||_{L^1(rdr)}||f||_{L^1(\mathbf{R})}. \tag{13}$$

To prove (13), we shall apply the results of the previous section. Notice first that we can decompose G as

$$\begin{aligned} G(u) &= \sum_{l=0}^{\infty}\int_{|u|(1+2^{-l-1})}^{|u|(1+2^{-l})} g(t)\frac{t^{1/2}}{(t-|u|)^{1/2}}dt + \int_{2|u|}^{\infty} g(t)\frac{t^{1/2}}{(t-|u|)^{1/2}}dt \\ &\leq C\sum_{l=0}^{\infty}\frac{2^{l/2}}{|u|}\int_{|u|}^{|u|(1+2^{-l})} g(t)tdt + \int_{|u|}^{\infty} g(t)dt = C\sum_{l=0}^{\infty}2^{-l/2}G_l(u) + h(u), \end{aligned}$$

where $h(u) = \int_{|u|}^{\infty} g(t)dt$, and

$$\begin{aligned} G_l(u) &= \frac{1}{2^{-l}|u|}\int_{|u|}^{|u|(1+2^{-l})} g(t)tdt \\ &\leq \sum_{k\in\mathbf{Z}}\left(\int_{(1+2^{-l})^{-k}}^{(1+2^{-l})^{2-k}} g(t)tdt\right)2^l(1+2^{-l})^k\chi_{((1+2^{-l})^{-k},(1+2^{-l})^{1-k}]}(|u|). \end{aligned}$$

Observe that h is even and decreasing in $(0,\infty)$ and that $||h||_{L^1(\mathbf{R})} = 2||g||_{L^1(rdr)}$. Also,

$$\sum_{k\in\mathbf{Z}}\left[\int_{(1+2^{-l})^{-k}}^{(1+2^{-l})^{2-k}} g(t)tdt\right] = 2||g||_{L^1(rdr)}.$$

Therefore, we obtain from Theorem 2 the estimate

$$\left\|\frac{1}{r}G_l(r.)*f(\theta)\right\|_{L^{1,\infty}(\mathbf{R}^2_+, rd\theta dr)} \leq C(1+l)^2||g||_{L^1(rdr)}||f||_{L^1(\mathbf{R})}.$$

Theorem 1 gives a similar estimate for $\frac{1}{r}h(r.)*f$. Now, inequality (13) follows from an application of the adding-up lemma of Stein and N. Weiss. ∎

4. Further remarks and some open questions

If one tries to prove the boundedness of M_Ω^* on radial functions in higher dimensions, there are two main obstacles one has to overcome. First, to extend Theorems 1 and 2 to functions in $L^1(\mathbf{R}^n)$ and, second, to bypass the lack of group structure in $\mathbf{S}^n$, $n > 1$. The group structure of $\mathbf{S}^1$ was essential for the case treated. The extension of Theorem 1 is described in the following result:

THEOREM 4: *Let h be a positive, radial and radially decreasing function in $L^1(\mathbf{R}^n)$. For $f \in L^1(\mathbf{R}^n)$, $x \in \mathbf{R}^n$, $y > 0$, we define $Tf(x,y) = \frac{1}{y}h(y.) * f(x)$. Then, T is a bounded operator from $L^1(\mathbf{R}^n)$ into $L^{1,\infty}(\mathbf{R}_+^{n+1}, y^n dxdy)$, with a constant depending only on $||h||_1$.*

Proof: The main steps of the proof of this theorem are similar to those given in the proof of the case $n = 1$. We sketch some of the modifications needed here.

First, we can assume that h is of the form

$$h = \sum_{k\geq 0} 2^{kn} a_k \chi_{(-2^{-k-3}, 2^{-k-3}]^n}, \quad a_k \geq 0.$$

Next, we define for $l \in \mathbf{Z}$, $i \in \mathbf{Z}^n$ the cubes $Q_l^i = 2^{-l}i + (0, 2^{-l}]^n$. The conditional expectation at level l, in $\mathbf{R}^n$, is

$$E_l g = \sum_{i\in Z^n} \left(\frac{1}{|Q_l^i|} \int_{Q_l^i} g \right) \chi_{Q_l^i}.$$

Finally, an iteration of the claim in Section 2 shows that for $j \geq j_0$

$$h_{2^j} * f(x) \leq \sum_{\tau\in\{0,1\}^n} \sum_{k\geq 0} a_k E_{k+j} f_\tau \left(x + \frac{2^{-j_0}}{3}\tau \right),$$

where $f_\tau(u) = f(u - \frac{2^{-j_0}}{3}\tau)$. The rest is easy. We leave the details to the interested reader.

Extensions of Theorems 2 and 3 to higher dimensions will be considered in a forthcoming paper.

In the case $n = 1$, Theorem 1 and, consequently, Theorem 2, admit some generalizations of a different kind. Let us consider, for $\alpha \in \mathbf{R}$ and h even, positive, and decreasing on $(0, \infty)$, the operator

$$H_\alpha f(x,y) = \frac{1}{y^\alpha}(yh(y.) * f(x)).$$

The problem consists in describing the values of α for which we have the weak type estimate

$$(14)\qquad \int_0^\infty |\{x \in \mathbf{R}: |H_\alpha f(x,y)| > \lambda\}| y^\alpha \frac{dy}{y} \le \frac{1}{\lambda} ||h||_1\, ||f||_1.$$

Theorem 1 answers the question for $\alpha = 2$, and a quick look at its proof shows that the case $\alpha > 1$ follows from the same techniques.

Via the transformation $y^{-1} = t$, (14) is related to earlier work [Sj1] and [Sj2] by the first author. Theorem 1 of [Sj2] implies (14) for $\alpha < 0$, under the additional assumption that $\int h(x)| \log |x||dx < \infty$. Recently, T. Menarguez and the second author have weakened in [M-S] this integrability condition on h to $\int h(x)| \log | \log |x||| < \infty$. Counterexamples to (14) in the case $0 \le \alpha \le 1$ are given in [Sj1, pp. 229 and 248]. We do not know whether (14) holds for $\alpha < 0$ and general h.

It would be interesting to know whether in Theorem 2 one can take C_ϵ independent of ϵ. This is equivalent to the inequality

$$\left\| \frac{1}{y} G(y.) * f(x) \right\|_{L^{1,\infty}(\mu)} \le C ||G||_1\, ||f||_1,$$

with no restriction on the integrable functions G and f. As before, $d\mu = y dx dy$ in $\mathbf{R}^2_+$. There is a counterexample to the stronger inequality

$$\sum_j |\{x: 2^{-j} G(2^j.) * f(x) > J\lambda\}| 2^{2j} \le C \frac{1}{\lambda} ||G||_1\, ||f||_1, \quad \lambda > 0.$$

Indeed, let N be a large natural number. As G we choose the characteristic function of the set

$$\bigcup_{k=1}^{2^{2N}} [k2^N, k2^N + 1].$$

We let f be the measure $f = 2^{-2N} \sum_{1 \le k \le 2^{2N}} \delta_k$. If $R = 2^j$, $1 \le j \le N$, then $G(R.)$ is the characteristic function of

$$\bigcup_{k=1}^{2^{2N}} [k2^{N-j}, k2^{N-j} + 2^{-j}].$$

In the set

$$\bigcup_{k=2^{3N-j-1}}^{2^{3N-j}} [k, k + 2^{-j}],$$

one finds that $R^{-1}G(R.) * f \sim 2^{-N}$. Thus,

$$\sum_{j=1}^{N} |\{x \colon 2^{-j}G(2^j.) * f(x) > J2^{-N}/C\}|2^{2j} \geq \sum_{j=1}^{N} 2^{3N-2j}2^{2j}/C \geq N2^{3N}.$$

As $N \to \infty$, this is much larger than $2^N ||G||_1 \, ||f||_1 = 2^{3N}$.

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